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A Mathematical Analysis of the PML Method[1]

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A detailed mathematical analysis of the Berenger PML method for the electromagnetic equations is carried out on the PDE level, as well as for the semidiscrete and fully discrete formulations. It is shown that the split set of equations is not strongly well-posed and that under certain conditions its solutions may be inappropriate.

I. INTRODUCTION

A long standing problem in computational electromagnetics has been the issue of finding infinite space solutions on a finite numerical domain (see [4] for references). One way of preventing outgoing waves from reflecting from the artificial numerical boundaries is to introduce an absorbing layer rather than look for more efficient nonreflecting boundary conditions. Berenger (1994) introduced his PML ("perfectly matched layer") method in which the usual Maxwell equations are split, thus creating, in the absorbing layer, waves which decay in all directions of propagation and which match the internal vacuum solution. There are many reports [4] of successful application of his PML methodology, lowering the overall reflection coefficients by many orders of magnitude. Recently the PML approach has been applied to the field of acoustics [5]. In this study it was found that filtering was necessary in order to avoid temporal instabilities.

In this paper we conduct a detailed mathematical analysis of the PML formulation applied to the two-dimensional transverse-electric mode (TE) of Maxwell's equations. The conclusions of this analysis hold also for all other forms of the equations.

We were interested in the well-posedness of the PML formulation, and therefore investigated the pure-initial value problem. The main conclusion is that the PML split-form of Maxwell's equation is only *weakly well-posed* and therefore its solution diverges under some small perturbations, an example of which is provided.

Section II of this paper demonstrates that unlike the original Maxwell's equations, the PML split equations are *not strongly well-posed.* As a result, under a perturbation of size δ, the solution will have an explosive mode proportional to $\exp[\sqrt{\omega\delta/\mu_0 t}]$.

In Section III we consider the semidiscrete (discretizing space only) version of the PML formulation. The analysis is carried out for central spatial differencing of *arbitrary* order of accuracy. We show that each of the split (nonphysical) magnetic components diverges as the spatial mesh becomes finer ($\Delta x \to 0$), and for a fixed Δx they grow linearly in time.

In Section IV we analyze the widely used, fully discrete Yee scheme [4]. We show that the Yee algorithm applied to the split equations satisfies the von Neumann condition under the same time-step restrictions as when applied to Maxwell's equations. However, the von Neumann condition is only necessary and not always sufficient. This is true in the present case, where we show that the norm of nth power of the amplification matrix, G^n, grows linearly with n, i.e., $\|G^n\| \sim n$.

The findings of this investigation imply that under certain conditions, perhaps not always met in practical computations, such as a highly refined grid or very long time of integration, the PML method might not yield the appropriate results.

In a future paper we shall introduce a set of equations which are strongly well-posed and whose solution decays in all possible directions of wave propagation while being perfectly matched at the interface.

II. THE PML METHOD IN 2D—THE PDE LEVEL

We can study the mathematical consequences of the PML method by considering the two dimensional transverse-electric mode (TE) case [4]. Maxwell's equations in 2D for the TE case may be written as

$$\frac{\partial \mathbf{W}}{\partial t} = A\frac{\partial \mathbf{W}}{\partial x} + B\frac{\partial \mathbf{W}}{\partial y} + C\mathbf{W}, \qquad (2.1)$$

where $\mathbf{W} = (E_x, E_y, H_z)^{\mathrm{T}}$; E_x, E_y, and H_z being respectively the electric-field components in the x, y directions and the

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magnetic field component normal to the domain of calculation.

The matrix coefficients A, B, and C are given by

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\varepsilon_0} \\ 0 & -\frac{1}{\mu_0} & 0 \end{pmatrix}; \quad B = \begin{pmatrix} 0 & 0 & \frac{1}{\varepsilon_0} \\ 0 & 0 & 0 \\ \frac{1}{\mu_0} & 0 & 0 \end{pmatrix}; \tag{2.2}$$

$$C = \begin{pmatrix} -\frac{\sigma}{\varepsilon_0} & 0 & 0 \\ 0 & -\frac{\sigma}{\varepsilon_0} & 0 \\ 0 & 0 & -\frac{\sigma^*}{\mu_0} \end{pmatrix}$$

Here ε_0 and μ_0 are the free space permittivity and permeability, and σ and σ^* denote, respectively, possible electric and magnetic losses assigned to free space. The speed of light in free space is given by $c = (\varepsilon_0\mu_0)^{-1/2}$. Note that in the rest of this paper we will use $H = H_z$ without introducing any ambiguity.

The system (2.1) can be symmetrized through the following change of variables: Let

$$\mathbf{V} = T_0^{-1}\mathbf{W} = (\tilde{E}_x, \tilde{E}_y, \tilde{H}), \tag{2.3}$$

where

$$T_0^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{\frac{\mu_0}{\varepsilon_0}} \end{pmatrix}; \quad T_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{\frac{\varepsilon_0}{\mu_0}} \end{pmatrix}. \tag{2.4}$$

The resulting system is

$$\mathbf{V}_t = \tilde{A}\mathbf{V}_x + \tilde{B}\mathbf{V}_y + \tilde{C}V, \tag{2.5}$$

where

$$\tilde{A} = T_0^{-1}AT_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -c \\ 0 & -c & 0 \end{pmatrix},$$

$$\tilde{B} = T_0^{-1}BT_0 = \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ c & 0 & 0 \end{pmatrix}$$

and

$$\tilde{C} = T_0^{-1}CT_0 = C. \tag{2.6}$$

Also note that using the above, $\tilde{E}_x = E_x$, $\tilde{E}_y = E_y$, and $\tilde{H} = \sqrt{\mu_0/\varepsilon_0}H$, and we have

$$\mathbf{V} = (E_x, E_y, \sqrt{\mu_0/\varepsilon_0}H). \tag{2.7}$$

Since in (2.5) all the matrix coefficients are now symmetric, the system is by definition symmetric hyperbolic [3, p. 119], and *therefore it is strongly well-posed.*

Next we will show that the PML formulation is *not* strongly well-posed. The loss of this property is very significant in that it will lead to instabilities in any numerical scheme. This point will be elaborated further in the next section.

Under the PML-formulation the 3×3 system becomes the following 4×4 system, due to the splitting of the orthogonal magnetic field, H, into nonphysical components H_x and H_y. The system may be written as

$$\frac{\partial \mathbf{W}_b}{\partial t} = A_b\frac{\partial \mathbf{W}_b}{\partial x} + B_b\frac{\partial \mathbf{W}_b}{\partial y} + C_b\mathbf{W}_b, \tag{2.8}$$

where

$$\mathbf{W}_b = (E_x, E_y, H_x, H_y)^{\mathrm{T}}, \tag{2.9}$$

$$A_b = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\varepsilon_0} & -\frac{1}{\varepsilon_0} \\ 0 & -\frac{1}{\mu_0} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad B_b = \begin{pmatrix} 0 & 0 & \frac{1}{\varepsilon_0} & \frac{1}{\varepsilon_0} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{\mu_0} & 0 & 0 & 0 \end{pmatrix} \tag{2.10}$$

$$C_b = \begin{pmatrix} -\frac{\sigma_y}{\varepsilon_0} & 0 & 0 & 0 \\ 0 & -\frac{\sigma_x}{\varepsilon_0} & 0 & 0 \\ 0 & 0 & -\frac{\sigma_x^*}{\mu_0} & 0 \\ 0 & 0 & 0 & -\frac{\sigma_y^*}{\mu_0} \end{pmatrix}.$$

Note that the losses are not necessarily isotropic.

The lower order term, $C_b\mathbf{W}_b$, in (2.8) does not affect the well-posedness of the problem and we shall, therefore, study this issue by considering (2.8) without this term.

First we shall show that (2.8) cannot be simultaneously symmetrized by a similarity transformation. Consider the diagonalizer of A_b,

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & c\varepsilon_0 & -c\varepsilon_0 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \tag{2.11}$$

$$T^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2c\varepsilon_0} & \frac{1}{2c\varepsilon_0} \\ 0 & \frac{1}{2} & -\frac{1}{2c\varepsilon_0} & -\frac{1}{2c\varepsilon_0} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then we have

$$T^{-1}A_bT = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -c & 0 & 0 \\ 0 & 0 & -c & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.12}$$

The most general diagonalizer of A_b is $S = TR$, where the columns of T are the eigenvectors of A_b and R is a matrix such that the column of S are the most general representation of the eigenvectors of A_b. In our case, the first and fourth columns of T correspond to a zero eigenvalue. Therefore the first and fourth columns of S are linear combinations of the corresponding eigenvectors in T. Thus,

$$R = \begin{pmatrix} \alpha & 0 & 0 & \beta \\ 0 & \tau & 0 & 0 \\ 0 & 0 & \Omega & 0 \\ \gamma & 0 & 0 & \delta \end{pmatrix}.$$

If S cannot symmetrize the matrix B_b, via a similarity transformation, then A_b and B_b cannot be symmetrized simultaneously [1]. Indeed,

$$S^{-1}B_bS = \begin{pmatrix} \frac{\alpha\beta\varepsilon_0c^2}{\Delta} & -\frac{\delta\tau c}{\Delta} & \frac{\delta\Omega c}{\Delta} & \frac{\beta^2\varepsilon_0c^2}{\Delta} \\ -\frac{\alpha c}{2\tau} & 0 & 0 & -\frac{\beta c}{2\tau} \\ \frac{\alpha c}{2\tau} & 0 & 0 & \frac{\beta c}{2\tau} \\ -\frac{\alpha^2\varepsilon_0c^2}{\Delta} & \frac{\gamma\tau c}{\Delta} & -\frac{\gamma\Omega c}{\Delta} & -\frac{\alpha\beta\varepsilon_0c^2}{\Delta} \end{pmatrix}, \tag{2.13}$$

where

$$\Delta = \alpha\delta - \beta\gamma.$$

The only way to make $(S^{-1}B_bS)_{1,4} = (S^{-1}B_bS)_{4,1}$, is by making $\alpha = \beta = 0$. This makes R, and therefore S, singular. Clearly A_b and B_b cannot be symmetrized simultaneously. Thus the PML equations (2.8) have lost an important property of the original TE equations (2.1), namely symmetry. This by itself does not imply non-well-posedness.

We will, however, show that the pure initial-value problem for (2.8) is only *weakly* well-posed. This means [3] that the norm of the solution in bounded (up to exponential growth in t) not only by the norm of the initial data, but also by the norm of the initial spatial derivatives. Such a weakly well-posed problem becomes ill-posed under some small perturbation.

Since (2.8), without the lower order term, is a 4×4 system with constant coefficients, we can examine its well-posedness by considering the system resulting from Fourier transforming the equations. The resulting system of equations is

$$\frac{\partial \hat{E}_x}{\partial t} = \frac{i\omega_2}{\varepsilon_0}(\hat{H}_x + \hat{H}_y) \tag{2.14a}$$

$$\frac{\partial \hat{E}_y}{\partial t} = -\frac{i\omega_1}{\varepsilon_0}(\hat{H}_x + \hat{H}_y) \tag{2.14b}$$

$$\frac{\partial \hat{H}_x}{\partial t} = -\frac{i\omega_1}{\mu_0}\hat{E}_y \tag{2.14c}$$

$$\frac{\partial \hat{H}_y}{\partial t} = \frac{i\omega_2}{\mu_0}\hat{E}_x, \tag{2.14d}$$

where the Fourier transform $\hat{F}$, of F, is defined by

$$F(x, y, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{\infty}^{\infty} \hat{F}(\omega_1, \omega_2, t)e^{i(\omega_1 x + \omega_2 y)}\, d\omega_1\, d\omega_2, \qquad (2.15)$$

and therefore, as usual, $i\omega_1$ and $i\omega_2$ represent partial differentiation in the x and y directions, respectively.

The initial values appropriate for (2.14) are

$$\hat{E}_x(0) = \hat{e}_0, \quad \hat{E}_y(0) = \hat{g}_0 \qquad (2.16)$$

$$\hat{H}_x(0) = \hat{h}_0 - \ell_0, \quad \hat{H}_y(0) = \ell_0, \qquad (2.17)$$

where (2.17) reflects the fact that for the original physical problem, (2.1)–(2.2), there is only one magnetic initial value, $\hat{H}(0) = \hat{H}_x(0) + \hat{H}_y(0) = \hat{h}_0$.

The solution of (2.14), subject to (2.16)–(2.17), is

$$\hat{E}_x = \frac{\omega_1^2}{\omega_1^2 + \omega_2^2}\hat{e}_0 + \frac{\omega_1\omega_2}{\omega_1^2 + \omega_2^2}\hat{g}_0 + \frac{i\omega_2}{\varepsilon_0 c\sqrt{\omega_1^2 + \omega_2^2}}(\hat{h}_0 \sin \nu t - \beta \cos \nu t) \qquad (2.18a)$$

$$\hat{E}_y = \frac{\omega_1\omega_2}{\omega_1^2 + \omega_2^2}\hat{e}_0 + \frac{\omega_2^2}{\omega_1^2 + \omega_2^2}\hat{g}_0 - \frac{i\omega_1}{\varepsilon_0 c\sqrt{\omega_1^2 + \omega_2^2}}(\hat{h}_0 \sin \nu t - \beta \cos \nu t) \qquad (2.18b)$$

$$\hat{H}_x = \frac{\omega_2^2\hat{h}_0 - (\omega_1^2 + \omega_2^2)\hat{\ell}_0}{\omega_1^2 + \omega_2^2} - \frac{i\omega_1\omega_2(\omega_1\hat{e}_0 + \omega_2\hat{g}_0)}{(\omega_1^2 + \omega_2^2)\mu_0}t + \frac{\omega_1^2}{\omega_1^2 + \omega_2^2}(\hat{h}_0 \cos \nu t + \omega \sin \nu t) \qquad (2.18c)$$

$$\hat{H}_y = -\frac{\omega_2^2\hat{h}_0 - (\omega_1^2 + \omega_2^2)\hat{\ell}_0}{\omega_1^2 + \omega_2^2} + \frac{i\omega_1\omega_2(\omega_1\hat{e}_0 + \omega_2\hat{g}_0)}{(\omega_1^2 + \omega_2^2)\mu_0}t + \frac{\omega_2^2}{\omega_1^2 + \omega_2^2}(\hat{h}_0 \cos \nu t + \beta \sin \nu t), \qquad (2.18d)$$

where

$$\nu = c\sqrt{\omega_1^2 + \omega_2^2}, \quad \beta = \frac{i\omega_2\hat{e}_0 - i\omega_1\hat{g}_0}{\mu_0 c\sqrt{\omega_1^2 + \omega_2^2}}. \qquad (2.19)$$

Note that while, from (2.18), we get

$$\hat{H} = \hat{H}_x + \hat{H}_y = h_0 \cos \nu t + \beta \sin \nu t$$

which is also the solution to the original 3×3 system, each (nonphysical) component, $\hat{H}_x$ and $\hat{H}_y$, has a linear growth in time with a coefficient which is not bounded in ω_1, ω_2. This implies that $\|H_x(t)\|$ and $\|H_y(t)\|$ cannot be bounded by the norm of the initial data, but rather depends also on the norms of the initial spatial derivatives of $\hat{E}_x$ and $\hat{E}_y$. *Thus the system (2.8) is only weakly well-posed.* It is well known [3] that unlike strongly well-posed problems, which remain strongly well-posed under general small perturbations, weakly well-posed systems become ill-posed under some small perturbations.

Next we shall show that the system (2.8) is ill-posed under the perturbation represented by

$$\begin{pmatrix} 0 & 0 & -\delta & \delta \\ 0 & 0 & -\delta & \delta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ H_x \\ H_y \end{pmatrix}. \qquad (2.20)$$

The perturbed (2.8), after Fourier transforming, becomes

$$\frac{\partial \hat{W}_b}{\partial t} = \hat{M}_b\hat{W}_b, \qquad (2.21)$$

where

$$\hat{M}_b = \begin{bmatrix} 0 & 0 & \dfrac{i\omega_2}{\varepsilon_0} - \delta & \dfrac{i\omega_2}{\varepsilon_0} + \delta \\ 0 & 0 & -\dfrac{i\omega_1}{\varepsilon_0} - \delta & -\dfrac{i\omega_1}{\varepsilon_0} + \delta \\ 0 & -\dfrac{i\omega_1}{\mu_0} & 0 & 0 \\ \dfrac{i\omega_2}{\mu_0} & 0 & 0 & 0 \end{bmatrix}. \qquad (2.22)$$

The eigenvalues of $\hat{M}_b$, λ, satisfy the quartic algebraic equation

$$\lambda^4 + \left[c^2(\omega_1^2 + \omega_2^2) - i\frac{(\omega_1 + \omega_2)}{\mu_0}\delta\right]\lambda^2 - \frac{2i}{\mu_0}c^2(\omega_1\omega_2)(\omega_1 + \omega_2)\delta = 0. \qquad (2.23)$$

For ill-posedness it is enough to show that at least one of the four λ's has a positive real part which grows with *any* combination of ω_1, ω_2. For example, consider the case of $\omega_1 = \omega_2 = \omega > 0$. Then (2.23) becomes

$$(\lambda^2 + 2c^2\omega^2)\left(\lambda^2 - \frac{2i\omega}{\mu_0}\delta\right) = 0. \qquad (2.24)$$

The first factor gives us the usual purely imaginary eigenvalues. The second factor yields

$$\lambda = (2i)^{1/2} \frac{\omega^{1/2}\delta^{1/2}}{\mu_0^{1/2}} = \pm(1+i)\left(\frac{\omega\delta}{\mu_0}\right)^{1/2}. \qquad (2.25)$$

Clearly one of the roots has a real positive part that grows as $\sqrt{\omega\delta/\mu_0}$; i.e., we have an ill-posed problem under small perturbation.

III. THE 2D PML METHOD—THE SEMI-DISCRETE LEVEL

We consider the right-hand side of (2.8) (again without the lower order term). Under symmetric differencing a first derivative operator of order of accuracy p may be written as

$$\frac{\partial}{\partial x} = \frac{1}{\Delta x} \sum_{m=-M_p}^{M_p} \sum_{\ell=-L_p}^{L_p} \alpha_{m\ell} S_x^m S_y^\ell + O(\Delta x^p, \Delta y^p). \qquad (3.1)$$

A similar expression holds for $\partial/\partial y$.

S_x and S_y are the shift operators in the x and y directions, respectively; $2M_p$ and $2L_p$ are the numerical domains of dependence. In addition, $\alpha_{m,\ell} = -\alpha_{-m,\ell}$, $\alpha_{m,\ell} = -\alpha_{m,-\ell}$. When (3.1) and the analogous expression for $\partial/\partial y$ are applied to a linear combination of waves made up of the components $e^{iqx}e^{iry}$ (where $x = j\,\Delta x$, $y = k\,\Delta y$), we get expressions of the form

$$\frac{\partial}{\partial x} \to iQ_1(\theta, \phi)\frac{1}{\Delta x}$$

$$\frac{\partial}{\partial y} \to iQ_2(\theta, \phi)\frac{1}{\Delta y},$$

where $\theta = q\,\Delta x$, $\phi = r\,\Delta x$, and Q_1, Q_2 are finite polynomials in $\sin m\theta$, $\sin \ell\phi$.

Thus the semidiscrete version of (2.8) (without the autonomous term) will take the form

$$\frac{\partial E_x}{\partial t} = \frac{iQ_2}{\varepsilon_0\,\Delta y}(H_x + H_y) \qquad (3.2a)$$

$$\frac{\partial E_y}{\partial t} = -\frac{iQ_1}{\varepsilon_0\,\Delta x}(H_x + H_y) \qquad (3.2b)$$

$$\frac{\partial H_x}{\partial t} = -\frac{iQ_1}{\mu_0\,\Delta x}E_y \qquad (3.2c)$$

$$\frac{\partial H_y}{\partial t} = \frac{iQ_2}{\mu_0\,\Delta y}E_x \qquad (3.2d)$$

with initial values (see Eqs. (2.16), (2.17)):

$$E_x(0) = e_0, \quad E_y(0) = g_0, \quad H_x = h_0 - \ell_0, \quad H_y = \ell_0. \qquad (3.3)$$

The system (3.2), (3.3) is identical to (2.14), (2.16), (2.17) with $\omega_1 \to Q_1/\Delta x$, $\omega_2 \to Q_2/\Delta y$, $\hat{e}_0 \to e_0$, $\hat{g}_0 \to g_0$, $\hat{h}_0 \to h_0$, and $\hat{\ell}_0 \to \ell_0$. We can therefore use (2.18) to write down the solution to (3.2). In particular we get for the "secular" terms H_x, H_y:

$$\begin{aligned} H_x &= \frac{Q_2^2}{Q_1^2+Q_2^2}h_0 - \ell_0 - \frac{iQ_1Q_2}{\mu_0}\left[\frac{\mathcal{R}Q_1e_0 + Q_2g_0}{\mathcal{R}^2Q_1^2+Q_2^2}\right]\frac{t}{\Delta x} \\ &\quad + \frac{\mathcal{R}^2Q_1^2}{\mathcal{R}^2Q_1^2+Q_2^2}[h_0\cos\nu t + \beta\sin\nu t] \\ H_y &= -\frac{Q_2^2}{Q_1^2+Q_2^2}h_0 + \ell_0 + \frac{iQ_1Q_2}{\mu_0}\left[\frac{\mathcal{R}Q_1e_0 + Q_2g_0}{\mathcal{R}^2Q_1^2+Q_2^2}\right]\frac{t}{\Delta x} \\ &\quad + \frac{Q_2^2}{\mathcal{R}^2Q_1^2+Q_2^2}[h_0\cos\nu t + \beta\sin\nu t], \end{aligned} \qquad (3.4)$$

where $\mathcal{R} = \Delta y/\Delta x$ is the "computational-cell aspect ratio." Since $Q_1(\theta, \phi)$ and $Q_2(\theta, \phi)$ are bounded we see that under mesh refinement (with fixed $\mathcal{R}$) H_x and H_y will separately diverge. Since in real computation there are always round-off errors in the sum $H = H_x + H_y$, the two growing terms which theoretically cancel each other, will not do so. This is true, according to (3.4) for any spatially nondissipative discretization, and also for any temporal finite differencing. A specific example for a commonly used algorithm is presented in the next section.

IV. THE APPLICATION OF YEE METHOD TO THE 2D PML FORMULATION

A popular scheme is the one due to Yee [4]. The algorithm is staggered both in space and time and is of second-order accuracy in both. For the TE case considered in the previous section the Yee formulation applied to the PML system gives

$$\begin{aligned} E_x^{n+1} &= E_x^n + \frac{\Delta t}{\varepsilon_0}\delta_y(H_x^{n+1/2} + H_y^{n+1/2}) \\ E_y^{n+1} &= E_y^n - \frac{\Delta t}{\varepsilon_0}\delta_x(H_x^{n+1/2} + H_y^{n+1/2}) \\ H_x^{n+1/2} &= H_x^{n-1/2} - \frac{\Delta t}{\mu_0}\delta_x E_y^n \\ H_y^{n+1/2} &= H_y^{n+1/2} + \frac{\Delta t}{\mu_0}\delta_y E_x^n, \end{aligned} \qquad (4.1)$$

where

$$\begin{aligned} \delta_x u_{j,k} &= \frac{1}{\Delta x}(u_{j+1,k} - u_{j-1,k}) \\ \delta_y u_{j,k} &= \frac{1}{\Delta y}(u_{j,k+1} - u_{j,k-1}). \end{aligned} \qquad (4.2)$$

Equation (4.1) may be cast in the following matrix form after setting

$$\Delta t \to \frac{\Delta t}{c}, \quad H_x, H_y \to \sqrt{\varepsilon_0/\mu_0}(H_x, H_y)$$

$$\begin{pmatrix} 1 & 0 & -\Delta t\delta_y & -\Delta t\delta_y \\ 0 & 1 & \Delta t\delta_x & \Delta t\delta_x \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E_x^{n+1} \\ E_y^{n+1} \\ H_x^{n+1/2} \\ H_y^{n+1/2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\Delta t\delta_x & 1 & 0 \\ \Delta t\delta_y & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} E_x^{n} \\ E_y^{n} \\ H_x^{n-1/2} \\ H_y^{n-1/2} \end{pmatrix}, \quad (4.3)$$

or more compactly,

$$G_L W_b^{n+1} = G_R W_b^n, \quad (4.4)$$

where the definitions of W_b^{n+1}, W_b^n, G_L, and G_R are self-evident.

The Fourier symbols of δ_x and δ_y are, respectively,

$$\delta_x \to i2\lambda_x \sin\theta = ik_1, \quad \delta_y \to i2\lambda_y \sin\phi = ik_2, \quad (4.5)$$

where $\lambda_x = \Delta t/\Delta x$, $\lambda_y = \Delta t/\Delta y$, and $-\pi/2 \le \theta, \phi \le \pi/2$.

The transformed system (4.4), after inverting, becomes

$$\hat{W}^{n+1} = \hat{G}_L^{-1}\hat{G}_R\hat{W}^n = \hat{G}\hat{W}^n, \quad (4.6)$$

where

$$G = \begin{pmatrix} 1 & 0 & ik_2 & ik_2 \\ 0 & 1 & -ik_1 & -ik_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -ik_1 & 1 & 0 \\ ik_2 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-k_2^2 & k_1k_2 & ik_2 & ik_2 \\ k_1k_2 & 1-k_1^2 & -ik_1 & -ik_1 \\ 0 & -ik_1 & 1 & 0 \\ ik_2 & 0 & 0 & 1 \end{pmatrix}. \quad (4.7)$$

A necessary, *but not sufficient,* condition for the stability of (4.6) is that all the eigenvalues of G are less than or equal to 1 in magnitude (the von Neumann condition). The eigenvalues μ of G are

$$\mu = 1, 1, 1 - \frac{\gamma}{2} \pm \sqrt{(\gamma/2)^2 - \gamma} = 1, 1, \mu_+, \mu_-, \quad (4.8)$$

where $\gamma = k_1^2 + k_2^2$. These values of μ satisfy the necessary condition for stability, provided we have $\gamma \le 4$, leading to the CFL condition $\lambda_x^2 + \lambda_y^2 \le 1$. Under this condition, all the μ's in (4.8) satisfy $|\mu| = 1$. However, as we shall shortly show, even though the scheme meets the von Neumann condition, it is *unconditionally* unstable. In fact $\|G^n\| \sim n$, where $n = t/\Delta t$ is the number of temporal iterations. This follows from the fact that the amplification matrix G may be written

$$G = T_J \begin{pmatrix} \mu_+ & 0 & 0 & 0 \\ 0 & \mu_- & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} T_J^{-1} = T_J J T_J^{-1}, \quad (4.9)$$

where T_J is the "Jordanizing" matrix

$$T_J = \begin{pmatrix} k_2 & k_2 & 0 & i/k_2 \\ -k_1 & -k_1 & 0 & i/k_1 \\ -\dfrac{ik_1^2}{1-\mu_+} & -\dfrac{ik_1^2}{1-\mu_-} & 1 & 0 \\ -\dfrac{ik_2^2}{1-\mu_+} & -\dfrac{ik_2^2}{1-\mu_-} & -1 & 0 \end{pmatrix}. \quad (4.10)$$

Note that

$$G^n = T_J J^n T_J^{-1} = T_J \begin{pmatrix} \mu_+^n & 0 & 0 & 0 \\ 0 & \mu_-^n & 0 & 0 \\ 0 & 0 & 1 & n \\ 0 & 0 & 0 & 1 \end{pmatrix} T_J^{-1}. \quad (4.11)$$

Using

$$\|G^n\| = \max_{\forall\alpha\neq 0} \frac{\|G^n\alpha\|}{\|\alpha\|}, \tag{4.12}$$

we have

$$\|G^n\| \geq \frac{\|G^n\alpha_0\|}{\|\alpha_0\|} \quad \forall\alpha_0. \tag{4.13}$$

In particular we pick

$$\alpha_0 = T_j(0, 0, 0, 1)^{\mathrm{T}} = (i/k_2, i/k_1, 0, 0). \tag{4.14}$$

Then

$$G^n\alpha_0 = (i/k_2, i/k_1, n, -n) \tag{4.15}$$

and, thus,

$$\begin{aligned}\|G^n\| &\geq \frac{\|G^n\alpha_0\|}{\|\alpha_0\|} = \frac{\sqrt{1/k_1^2 + 1/k_2^2 + 2n^2}}{\sqrt{1/k_1^1 + 1/k_2^2}} \\ &\geq \frac{\sqrt{2}n}{\sqrt{1/k_1^2 + 1/k_2^2}} \quad \forall k_1, k_2.\end{aligned}$$

Thus $\|G^n\|$ grows linearly with n for practically any wave length.

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